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DECOMPOSITIONS OF COMPLETE GRAPHS INTO REGULAR BICHROMATIC FACTORS

Anton KOTZIG

Centre de recherches mathématiques, Université de Montréal, Montréal 101, Canada

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Abstract. The purpose of this paper is to find a necessary and sufficient condition for the existence of a decomposition of a complete graph with given number of vertices into regular bichromatic factors and to answer the question what is the possible number of factors in such a decomposition.

Throughout this paper the word graph will mean a finite undirected graph without loops and multiple edges, having at least one edge. If G is a graph, then $V(G)$ (resp. $E(G)$) is the vertex set (resp. edge set) of G. By K_n we denote the complete graph with the vertex set $V(K_n) = \{1, 2, ..., n\}$. A graph G is a bichromatic graph if there exists a decomposition of $V(G)$ into two classes of vertices V' and V'' such that every edge of $E(G)$ joins two vertices belonging to different classes of $\{V', V''\}$. Then obviously it is possible to colour the vertices of G by two colours so that every edge of G joins two vertices having different colours (V' and V'' are then the sets of "chromatic equivalent" vertices). A graph G is regular iff all vertices of G have the same degree. A subgraph F of G is a factor of G iff $V(F) = V(G)$.

Lemma 1. *Let G be a regular bichromatic graph of degree n and let $\{V', V''\}$ be a decomposition of $V(G)$ into classes of chromatic equivalent vertices, then $|V'| = |V''|$.*

Proof. According to the definition of a bichromatic graph every edge of $E(G)$ joins a vertex of V' with a vertex of V''. Therefore, if we denote by $d_G(v)$ the degree of the vertex v in G, we have obviously:

$$n|V'| = \sum_{v \in V'} d_G(v) = \sum_{v \in V''} d_G(v) = n|V''|,$$

and as $n > 0$ we have $|V'| = |V''|$.

Corollary 1. *A regular bichromatic graph has an even number of vertices*

Lemma 2. *A decomposition of K_n into regular bichromatic factors exists if and only if n is even.*

Proof. Let $F = \{F_1, F_2, ..., F_r\}$ be a decomposition of K_n into bichromatic regular factors. Then $V(F_i) = V(K_n) = n$, and as F_i is a bichromatic regular graph, n must be even. It is well known that if n is even, then K_n can be decomposed into $n-1$ linear factors; but, since every linear graph is bichromatic and regular, the validity of the lemma follows directly.

Remark 1. The maximal number of bichromatic regular factors in a decomposition of K_n is clearly $n-1$ (and this decomposition exists, according to lemma 2, only if n is even). The question regarding the minimal number of factors in a decomposition with the required property will be answered in the following theorem.

Theorem 1. *Let k and n be natural numbers with the properties that $2^{k-1} < n \leq 2^k$ and n is even, then there exists a decomposition $F = \{F_1, F_2, ..., F_k\}$ of K_n into k regular bichromatic factors and k is the minimal number of factors in any decomposition of K_n into regular bichromatic factors*

Proof. It is clear that n must be even (see Lemma 2) and that the theorem is true for $n = 2$ (then $k = 1$ and $F = \{F_1\}$, where $F_1 = K_2$).

Suppose that the theorem is true for every k smaller than a natural number r. Let n be an even number such that $2^{r-1} < n \leq 2^r$. If $\frac{1}{2}n = m$ is also even, then we have $2^{r-2} < m \leq 2^{r-1}$, and according to our supposition, there exists a decomposition of K_m into $r-1$ regular bichromatic factors. Let K'_m (resp. K''_m) be the complete subgraph of K_n, where

$V(K'_m) = \{1, 2, ..., m\}$ and $V(K''_m) = \{m+1, m+2, ..., n\}$ and let $F' = \{F'_1, F'_2, ..., F'_{r-1}\}$ (resp. $F'' = \{F''_1, F''_2, ..., F''_{r-1}\}$) be a decomposition of K'_m (resp. of K''_m) into regular bichromatic factors such that the following holds: the edge of K''_m with endpoints $m+i$ and $m+j$ belongs to F''_x iff the edge of K'_m with endpoints i and j belongs to F'_x. Then obviously for every $x = 1, 2, ..., r-1$, $F_x = F'_x \cup F''_x$ is a regular bichromatic factor of K_n and the factor F_r of K_n containing all edges not belonging to $K'_m \cup K''_m$ is also a regular bichromatic factor of K_n. Therefore, if m is even, then there exists a decomposition $F = \{F_1, F_2, ..., F_r\}$ of K_n into r bichromatic regular factors.

Now let m be odd. Then $m+1$ is even and we have obviously $2^{r-2} < m+1 \leq 2^{r-1}$. According to our supposition there exists a decomposition $F^0 = \{F^0_1, F^0_2, ..., F^0_{r-1}\}$ of K_{m+1} into $r-1$ regular bichromatic factors. Let us have again $V(K'_m) = \{1, 2, ..., m\}$ and $V(K''_m) = \{m+1, m+2, ..., n\}$ and let us define the decomposition $\{G'_1, G'_2, ..., G'_{r-1}\}$ of K'_m (resp. the decomposition $G'' = \{G''_1, G''_2, ..., G''_{r-1}\}$ of K''_m) into bichromatic (not necessarily regular) factors as follows: the edge of K'_m with the endpoints i and j belongs to G'_x and the edge of K''_m with the endpoints $m+i$ and $m+j$ belongs to G''_x iff the edge of K_{m+1} with the endpoints i and j belongs to F^0_x. Denote by e^0_v ($v = 1, 2, ..., m$) the edge of K_{m+1} joining the vertices v and $m+1$ in K_{m+1} and by e_v the edge of K_n joining v and $m+v$ in K_n. Define for every $x = 1, 2, ..., r-1$ the edge set $H_x \subset \{e_1, e_2, ..., e_m\} = E \subset E(K_n)$ as follows: e_v belongs to H_x iff e^0_x belongs to F^0_x. If we put, for $x = 1, 2, ..., r-1$, $F_x = G'_x \cup G''_x \cup H_x$, then F_x is a regular bichromatic factor of K_n (and has the same degree as F^0_x) and the factor F_r of K_n containing all the edges not belonging to $F_1 \cup F_2 \cup ... \cup F_{r-1}$ is also a regular bichromatic graph (it is a so-called parity graph of degree $m-1$). Therefore $F = \{F_1, F_2, ..., F_r\}$ is a decomposition of K_n into r regular bichromatic factors and we have in both cases (m is even or odd) the following result: If n is an even number > 0 such that $2^{k-1} < n \leq 2^k$, then K_n can be decomposed into k regular bichromatic factors.

Now let us prove that k is the minimal number of factors in a decomposition with the required properties.

Let $D = \{D_1, D_2, ..., D_s\}$ be a decomposition of K_n into (not necessarily regular) bichromatic factors and let $W_i = \{V_i(0), V_i(1)\}$ be a decomposition of $V(D_i) = \{1, 2, ..., n\}$ into two classes of vertices $V_i(0)$ and

$V_i(1)$ such that every edge of D_i joins a vertex of $V_i(0)$ with a vertex of $V_i(1)$. Define the number $c_i(v)$ for every $v \in \{1, 2, ..., s\}$ as follows: $[v \in V_i(x) \in W_i] \Leftrightarrow [c_i(v) = x]$ and put

$$c(v) = \sum_{i=1}^{s} c_i(v) \cdot 2^{i-1} .$$

It is obvious that for every $v \in V(K_n) = \{1, 2, ..., n\}$, $c(v)$ belongs to the set $\{0, 1, 2, ..., 2^s - 1\} = S (\Rightarrow |S| = 2^s)$ and further $[u \neq v;\ \{u, v\} \subset \{1, 2, ..., n\}] \Rightarrow [c(u) \neq c(v)]$. In other words: the vertices of K_n can be coloured by at most 2^s colours so that every two adjacent vertices have different colours in this colouring. But then $2^s \geqslant n$ and from this the validity of the theorem follows immediately.

Theorem 2. *Let k and n be natural numbers with the property that $2^{k-1} < n \leqslant 2^k$. Then there exists a decomposition of K_n into k bichromatic factors and k is the minimal number of factors in any decomposition of K_n into bichromatic factors.*

Proof. The validity of this theorem follows from the validity of Theorem 1 [Immediately if n is even, while if n is odd then we obtain a decomposition of K_n from a decomposition of K_{n+1} after deleting the vertex denoted by $n+1$. We recall that in the proof of the fact that k is the minimal number of bichromatic factors in a decomposition of K_n into bichromatic factors, we did not suppose that the D_i's are regular.]

Theorem 3. *Let k, r and n be natural numbers with the following properties: (1) n is even; (2) $2^{k-1} < n \leqslant 2^k$; (3) $k \leqslant r \leqslant n-1$. Then there exists a decomposition of K_n into r regular bichromatic factors.*

Proof. According to Theorem 1 in our case there exists a decomposition $F = \{F_1, F_2, ..., F_k\}$ of K_n into k regular bichromatic factors. It is well-known that every bichromatic regular graph of degree d can be decomposed into d linear factors.

Let d_i be the degree of F_i and let $D_i = \{L_i(1), L_i(2), ..., L_i(\alpha_i)\}$ be a decomposition of F_i into linear factors of F_i. It is obvious that a linear factor of D_i is also a linear factor of K_n and that every composition of linear factors of D_i is a regular bichromatic factor of K_n. In this way we can obtain from F_i a decomposition of F_i into r_i regular bichromatic factors (each of which being also a regular bichromatic factor of K_n) for every r_i such that $1 \leq r_i \leq d_i$. Therefore K_n can be decomposed into r regular bichromatic factors for every r for which we have $r = r_1 + r_2 + \ldots + r_k$ and $1 \leq r_i \leq d_i$ $(i = 1, 2, \ldots, k)$. Hence the decomposition of K_n with the required properties exists for every r for which we have $k \leq r \leq n - 1$ (because $d_1 + d_2 + \ldots + d_k = n - 1$). The validity of other assertions of Theorem 3 is obvious directly from Theorem 1.

Remark 2. If the factors of a decomposition of K_n must be bichromatic but need not be regular, then the number of factors can be obviously greater than $n - 1$, but this number equals at most the number $\binom{n}{2}$ of edges of K_n. The decomposition $F = \{F_1, F_2, \ldots, F_r\}$ of K_n into $r = \binom{n}{2}$ bichromatic factors can be defined in the following way: every factor F_i contains exactly one edge of K_n. Therefore a generalization of Theorem 3 for the case of an odd n is of no interest.